Linear sytem theory – lecture 10

Ph. Müllhaupt

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Outline : Stability and SI-state feedback.

1. BIBO stability
2. Internal stability

   stability in the sense of Lyapunov

3. State feedback, single input
4. Eigenvalue assignment using canonical forms
5. " " " a Lyapunov Equation

1. Stability, BIBO

Def: BIBO stability

$$\left.\begin{array}{l} x(t_0) = 0 \\ u(t), \quad t \geq 0 \end{array}\right\}$$

$$M \quad \|u\| \leq M$$

$$\forall u \quad \Longrightarrow \quad \|y\| < C$$

$$\exists C$$

$$y(t) = \int_0^t g(\tau)\, u(t-\tau)\, d\tau.$$

Thm:

$$\int_0^\infty |g(\tau)|\, d\tau < M$$

$$\Longleftrightarrow$$

BIBO stability.

$$\{\delta(t)\} \longrightarrow \boxed{\quad} \longrightarrow \{g(t)\}$$

impulse response

$$|y(t)| = \left| \int_0^t g(\tau)\, u(t-\tau)\, d\tau \right|$$

$$\leq \int_0^t |g(\tau)|\, |u(t-\tau)|\, d\tau$$

$$\leq U_m \int_0^t |g(\tau)|\, d\tau$$

$(\exists U_m,\ |U(t)| \leq U_m < \infty)$

$$U_m \int_0^t |g(\tau)|\, d\tau \leq U_m M$$

$$\int_0^\infty |g(\tau)|\, d\tau < M \Longrightarrow \text{BIBO stability}$$

$$\text{BIBO} \Longrightarrow \int_0^\infty |g(\tau)|\, d\tau < M$$

$$A \Longrightarrow B$$

$$\overline{B} \Longrightarrow \overline{A}$$

$$\forall N \Rightarrow \exists t_i(N) \text{ s.t } \int_0^{t_i} |g(\tau)|\, d\tau \geq N$$

Let us choose $u(t_1 - \tau) = \begin{cases} 1 & \text{if } g(\tau) \geq 0 \\ -1 & \text{if } g(\tau) \leq 0 \end{cases}$

u is clearly bounded.

$$y(t_1) = \int_0^{t_1} g(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_0^{t_1} |g(\tau)|\, d\tau \geq N$$

Because $y(t_1)$ can be arbitrarily large, a bounded input can excite an unbounded output.

<u>SISO</u> Relation with the transfer function

$$G(s) = \mathcal{L}(\{g(t)\})$$

Decomposition of $G(s)$ in simple elements (1st order)

$$\frac{1}{s+p_i}, \frac{1}{(s+p_i)^2}, \ldots, \frac{1}{(s+p_i)^{m_i}}$$

$p_i \in \mathbb{C}$

$$\updownarrow$$

$$e^{p_i t}, \quad t e^{p_i t}, \quad \ldots, \quad t^{m_i - 1} e^{p_i t}$$

$$\boxed{\begin{gathered}\text{BIBO stable} \\ \Longleftrightarrow \\ Re(p_i) < 0\end{gathered}}$$

i.e., $\displaystyle\int_0^{\infty} \left| e^{p_i \tau} \right| d\tau < M \Longleftrightarrow Re(p_i) < 0$

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although $\dot{x} = u$ has a single pole at 0, it is still not BIBO stable since the input $u(t) = 1$ gives an unbounded response.

2. Internal stability

$$\dot{x} = Ax$$

$\mathcal{X}(x_0, t) = e^{At} x_0$
the solution trajectory.

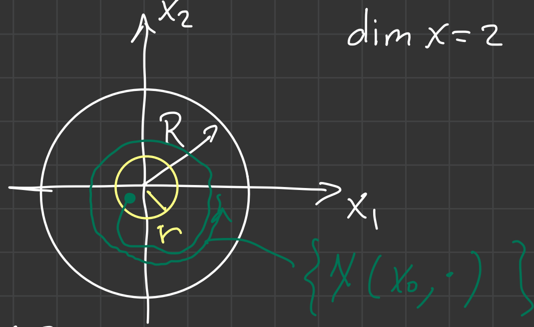
Def: Lyapunov stability

dim $x = 2$

$\forall R, \; \exists r(R),$

$\forall x_0, \; \|x_0\| < r,$

$\Longrightarrow \|\mathcal{X}(x_0, t)\| < R, \; \forall t.$



$\dot{x} = f(x)$ dim $x = n$.

Lyapunov: $V(x)$ continuous, $\mathbb{R}^n \mapsto \mathbb{R}$,

$V > 0, \; \forall x \neq 0$

$\dot{V} \leq 0 \; \Longrightarrow$ Lyapunov stability.

For linear systems: V can be defined by a positive definite matrix $P > 0$

$$V = \frac{1}{2} x^T P x$$

$$\dot{V} = \frac{1}{2} \dot{x}^T P x + \frac{1}{2} x^T P \dot{x}$$

$\dot{x} = Ax$

$$= \frac{1}{2} x^T A^T P x + \frac{1}{2} x^T P A x$$

$$= \frac{1}{2} x^T \left(A^T P + P A \right) x$$

Let us enforce $\dot{V} \leq 0$, hence

$$A^T P + P A \leq 0$$

$$\Downarrow$$

Lyapunov stable

<u>Asymptotic stability:</u>

$\dot{V} < 0, \forall x \neq 0 \implies$ asymptotic stability

$$\exists r_0, \forall x_0 \, \|x_0\| < r_0, \forall \lim_{t \to \infty} X(x_0, t) \to 0$$

$A^T P + P A < 0 \implies$ asymptotic stability + stability in the Lyapunov sense

(negative definite matrix)

Lyapunov matrix equation

$$\boxed{A^T P + PA = -Q}$$

$Q > 0$
(asymp. stab.)

$Q \geq 0$
(Lyap. stability)

<u>Thm</u>: All eigenvalues of A have a negative real part

$\iff$

<u>for any</u> $Q > 0$, the Lyapunov equation

$$A^T P + PA = -Q$$

has a unique positive definite solution $P > 0$.

<u>Necessity</u>: if A is stable and A has no two eigenvalues such that $\lambda_j + \lambda_i = 0$ the Lyapunov equation is nonsingular and has a unique solution P for any Q.

the solution can be expressed

$$\boxed{P = \int_0^\infty e^{A^T t} Q e^{At} dt}$$

$$A^T P + PA = \int_0^\infty A^T e^{A^T t} Q e^{At} dt$$

$(Q = Q^T)$

$$+ \int_0^\infty e^{A^T t} Q e^{At} A \, dt$$

$$= \int_0^\infty \frac{d}{dt}\left(e^{A^T t} Q e^{At}\right) dt$$

$$= e^{A^T t} Q e^{At} \Big|_0^\infty$$

$$0 - Q = -Q$$

Let us decompose Q as $\bar{Q}^T \bar{Q}$

$$x^T P x = \int_0^\infty x^T e^{A^T t} \bar{Q}^T \bar{Q} e^{At} x \, dt$$

$$= \int_0^\infty \| \bar{Q} e^{At} x \|_2^2 \, dt$$

Because both $\bar{Q}$ and e^{At} are nonsingular for any nonzero x, the integrand is positive for every t, thus $x^T P x$ is positive for any $x \neq 0$.

Sufficiency $Q > 0 \quad P > 0$

$$\Longrightarrow$$

A stable

$v \neq 0 \qquad Av = \lambda v$

$$v^* A^* = \lambda^* v^*$$

$$-v^* Q v = v^* \left(A^T P v + P A v \right)$$

$$= (\lambda^* + \lambda)\, v^* P v$$

$$2\, \mathrm{Re}(\lambda)\, v^* P v$$

since both $Q > 0$, $P > 0$
this implies $\mathrm{Re}(\lambda) < 0$

General Lyapunov equation

$$AM + MB = C$$

A : $n \times n$ $\quad$ C : $n \times m$

B : $m \times n$ $\quad$ M : $n \times m$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \end{bmatrix} + \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

Define $\mathcal{A}(M) = AM + MB$

$$\mathcal{A}(M) = C$$

maps nm dimensional space into itself

Eigenvalue of $\mathcal{A}(M)$:

$$\mathcal{A}(M) = \eta M$$

it turns out $\eta_k = \lambda_i + \mu_j$

λ_i $1, 2, \dots, n$: eig. of A.

μ_j $1, 2, \dots, m$: eig. of B.

Let u_i be the right eigenvector of A associated with λ_i

$$A u_i = \lambda_i u_i$$

Let v_j be the left eigenvector of B assoc. with μ_j

$$v_j B = \mu_j v_j$$

$$\mathcal{A}(u_i v_j) = A u_i v_j + u_i v_j B$$

$$= (\lambda_i + \mu_j) u_i v_j$$

Now for the Lyapunov equation A and A^T have the same eigenvalues.

$$A^T P + PA = \mathcal{A}(P)$$

correspondence: $A^T \longrightarrow A$

$A \longrightarrow B$

$$A^T u_i = \lambda_i u_i$$

$$v_j A = \mu_j v_j$$

$$\mathcal{A}(u_i v_j) = A^T u_i v_j + u_i v_j A$$

⚠ $\underline{\mu_j = \lambda_j}$

$$= \lambda_j u_i v_j + \mu_j u_i v_j = (\lambda_j + \lambda_i) u_i v_j$$

⚠ careful when $\lambda_j + \lambda_i = 0$.

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3. <u>State feedback</u> single input.

$$\dot{x} = Ax + Bu \qquad y = cx$$

$$u = -kx + r$$

$$\dot{x} = Ax - kbx + br$$

$$\dot{x} = (A - bk)x + br \qquad y = cx$$

Thm: The pair $(A-bk, b)$, for any real $1 \times n$ constant vector k, is controllable, if and only if, (A, b) is controllable.

$$\mathcal{C} = \begin{bmatrix} b & Ab & A^2b & A^3b \end{bmatrix}$$

$$\mathcal{C}_f = \begin{bmatrix} b & (A-bk)b & (A-bk)^2b & (A-bk)^3b \end{bmatrix}$$

$$\mathcal{C}_f = \begin{bmatrix} b & Ab & A^2b & A^3b \end{bmatrix} \begin{bmatrix} 1 & -kb & -k(A-bk)b & -k(A-bk)^2b \\ 0 & 1 & -kb & -k(A-bk)b \\ 0 & 0 & 1 & -k(A-bk)b \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\square$

$$-bk(A-bk)b - Abkb + A^2b =$$

$$(-bk(A-bk) - Abk + A^2)b$$

$$(-bk(A-bk) + A(A-bk))b = (A-bk)^2b.$$

Since the matrix $\boxed{\,}$ is always nonsingular no matter the values in k, controllability is ensured if (A, b) is controllable.

However, observability is not invariant by state feedback

$$\dot{x} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} x$$

$$u = r - \begin{bmatrix} 3 & 1 \end{bmatrix} x$$

$$\dot{x} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} x$$

observability of transformed system

$$\overline{O} = \begin{bmatrix} c \\ c\overline{A} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$ not observable

prior to state feedback,

$$\mathcal{O} = \begin{bmatrix} 1 & 2 \\ 7 & 4 \end{bmatrix}$$ is observable

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Let change the char. pol. of an example

$$\dot{x} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$\Delta(s) = s^2 - 2s - 8 = (s-4)(s+2)$$

$$\left(\quad |sI - A| = \begin{vmatrix} s-1 & -3 \\ -3 & s-1 \end{vmatrix} = (s-1)^2 - 9 \quad \right)$$

one eigenvalue is unstable : 4

$$\dot{x} = \left(\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix} \right) x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$$

$$= \begin{bmatrix} 1-k_1 & 3-k_2 \\ 3 & 1 \end{bmatrix} x$$

$$\Delta_f(s) = s^2 + (k_1 - 2)s + 3(k_2 - k_1 - 8)$$

which can be used to assign the char. pol. to whatever value desired.

<u>Thm</u>

$$\Delta(s) = s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4$$

if $\dot{x} = Ax + bu$ is controllable,

then it can be transformed by

$$\bar{x} = Px$$

$$Q \triangleq P^{-1} = \begin{bmatrix} b & Ab & A^2b & A^3b \end{bmatrix} \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

into the controllable form

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{b}u =$$

$$\dot{\bar{x}} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u \qquad \text{I}$$

$$y = c\bar{x} = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix} \bar{x}$$

$$\bar{A} = PAP^{-1}$$

III. $$\bar{C} = \begin{bmatrix} 1 & -\alpha_1 & \alpha_1^2-\alpha_2 & -\alpha_1^3+2\alpha_1\alpha_2-\alpha_3 \\ 0 & 1 & -\alpha_1 & \alpha_1^2-\alpha_2 \\ 0 & 0 & 1 & -\alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

IV. $$\bar{C}^{-1} = \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

II. $$P = \bar{C} C^{-1}$$

$$Q \triangleq P^{-1} = C \bar{C}^{-1} =$$

V. $$\begin{bmatrix} b & Ab & A^2b & A^3b \end{bmatrix} \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The five steps I, II, III, IV, V for constructing the equivalence transformation.

We can use this to assign the poles

$$\Delta_f = s^4 + \bar{\alpha}_1 s^3 + \bar{\alpha}_2 s^2 + \bar{\alpha}_3 s + \bar{\alpha}_4$$

$$\dot{\bar{x}} = (\bar{A} - \bar{b}\bar{k})\bar{x} + \bar{b}r \qquad \bar{A} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\bar{\alpha}_1 & -\bar{\alpha}_2 & -\bar{\alpha}_3 & -\bar{\alpha}_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r$$

by comparing the char. pol., $\bar{k}$ can be chosen

$$\boxed{\bar{k} = \begin{bmatrix} \bar{\alpha}_1 - \alpha_1 & \bar{\alpha}_2 - \alpha_2 & \bar{\alpha}_3 - \alpha_3 & \bar{\alpha}_4 - \alpha_4 \end{bmatrix}}$$

$$u = r - kx = r - kP^{-1}\bar{x} = r - \bar{k}\bar{x}$$

$$\boxed{\bar{k} = kP^{-1}}$$

$A - bk$ and $\bar{A} - \bar{b}\bar{k}$ have the same eigenvalues and hence the same char. pol. (i.e. due to similarity transform).

$$s^4 + \bar{\alpha}_1 s^3 + \bar{\alpha}_2 s^2 + \bar{\alpha}_3 s + \bar{\alpha}_1$$

$$\bar{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{A} - \bar{b}\bar{k} = \begin{bmatrix} -\alpha_1 - \bar{k}_1 & -\alpha_2 - \bar{k}_2 & -\alpha_3 - \bar{k}_3 & -\alpha_4 - \bar{k}_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -\bar{\alpha}_1 & -\bar{\alpha}_2 & -\bar{\alpha}_3 & -\bar{\alpha}_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3, \bar{\alpha}_4$ are the corresponding coefficients of the desired eigenvalues.

$$k = \bar{k} P.$$

5. Eigenvalue assignment by solving the Lyapunov equation:

(A, b) controllable

a) select a matrix F $(n \times n)$ that has the selected eigenvalues.

b) select an arbitrary $1 \times n$ vector $\bar{k}$ such that $(F, \bar{k})$ observable

c) solve the unique T of the generalized Lyapunov equation (matlab lyap(a,b,c))

$$\boxed{AT - TF = b\bar{k}}$$

d) Compute the feedback gain

$$\boxed{k = \bar{k}\,T^{-1}}$$